Announcements

1) Webwork #2, 4

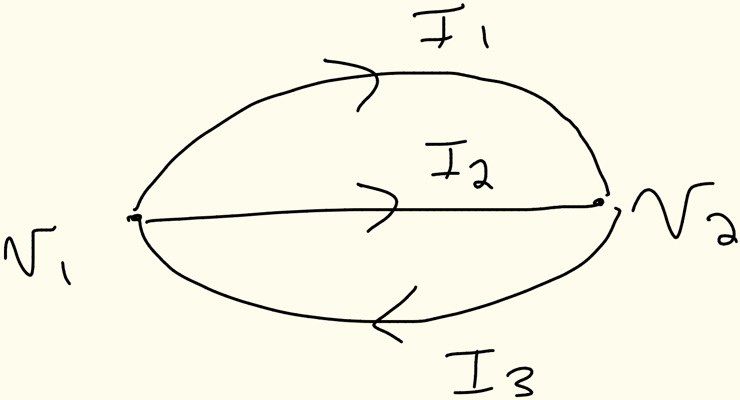
"perpendicular" = "orthogonal"

2) Math Colloquium today

3-4 in CB 2064

"p-adic numbers"

Recall Example 4 from last class



Edge-Node Incidence Matrix

$$A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}$$

Kirchoff's Current Law

- $I_1, I_2, \dots, I_m$ the currents along each edge (in amperes).
- $f_1, f_2, \dots, f_n$ external current sources associated to each node
- A = edge-node incidence matrix

$$\boxed{A^t \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_m \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}}$$

Current Law

What it says:

"The current flowing into any node is equal to the current flowing out of the node."

Note: no external current sources

$$f_1 = f_2 = \cdots = f_n = 0$$

for every node.

In that case, Kirschoff's Current Law becomes

$$A^t \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Example 4 Again

$$A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} \quad (3 \times 2)$$

$$A^t = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \quad (2 \times 3)$$

We assume no external current sources, so we get

$$\overbrace{\begin{bmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}}^{A^t} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

To solve, put the following matrix:

$$\begin{bmatrix} -1 & -1 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{bmatrix}$$

in rref.

We'd get

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This says

$$I_1 + I_2 - I_3 = 0,$$

so $\quad I_3 = I_1 + I_2,$

but we have infinitely many solutions depending on our choices for I_1, I_2.

Ohm's Law

$R_1, R_2, \ldots, R_m$ resistance along each edge (in Ohms).

$b_1, b_2, \ldots, b_m$ battery strength along each edge (in volts)

Let R be the $m \times m$ diagonal matrix

$$\begin{bmatrix} R_1 & 0 & \cdots & 0 \\ 0 & R_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & R_m \end{bmatrix} = R$$

Then

$$\boxed{R\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_m \end{bmatrix} + A\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}$$

Ohm's Law

Note: $v_1, v_2, \dots, v_n$ are now the electric potentials (voltage) at each node. We are assuming current flows from positive to negative.

If we write

$$I = \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_m \end{bmatrix} \text{ and}$$

$$V = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} - A \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix},$$

then subtracting $A \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

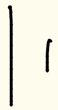
from both sides of Ohm's Law

gives $\boxed{RI = V}$

In words,

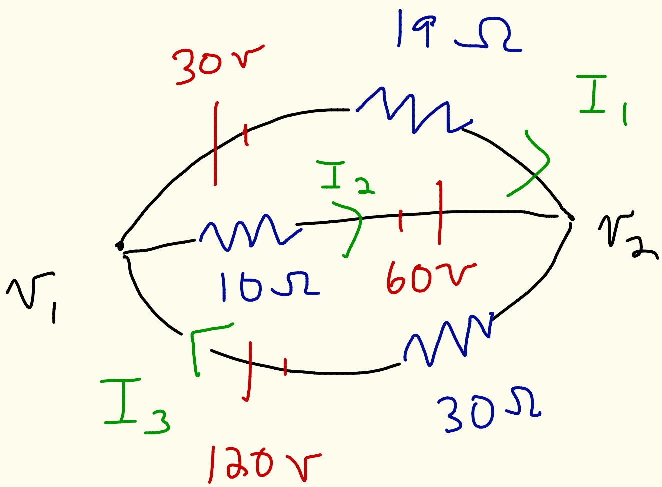
"Current times resistance is equal to the voltage".

Final Look at Example 4

Battery: 

small line is the positive end

Resistor: 



"Ω" = Ohms, "v" = volts

Ohm's Law

$$R\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} + A\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

I_1, I_2, I_3, v_1, v_2 are the unknowns.

$$R = \begin{bmatrix} 19 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 30 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$b_1 = -30$$ (since current flows the wrong way)

$$b_2 = 60$$

$$b_3 = 120$$

We then have

$$\begin{bmatrix} 19 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 30 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$= \begin{bmatrix} -30 \\ 60 \\ 120 \end{bmatrix}$$

How to use a single matrix to solve this problem?

We make a huge matrix that incorporates both laws:

$$\begin{bmatrix} \overbrace{R}^{m\times m} & \overbrace{A}^{m\times n} \\ \underbrace{A^t}_{n\times m} & \left(\text{bunch of zeros}\right) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_m \\ v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \\ f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

We use this to finish example 4

$R \qquad A$

$$\begin{bmatrix} 19 & 0 & 0 & -1 & 1 \\ 0 & 10 & 0 & -1 & 1 \\ 0 & 0 & 30 & 1 & -1 \\ -1 & -1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ v_1 \\ v_2 \end{bmatrix}$$

A^t

$$= \begin{bmatrix} -30 \\ 60 \\ 120 \\ 6 \\ 0 \end{bmatrix}$$

The solution comes from row-reducing

$$\begin{bmatrix} 19 & 0 & 0 & -1 & 1 & -30 \\ 0 & 10 & 0 & -1 & 1 & 60 \\ 0 & 0 & 30 & 1 & -1 & 120 \\ -1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \end{bmatrix}$$

rref is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -90/53 \\ 0 & 1 & 0 & 0 & 0 & 306/53 \\ 0 & 0 & 1 & 0 & 0 & 216/53 \\ 0 & 0 & 0 & 1 & -1 & -120/53 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We get

$$I_1 = -90/53$$

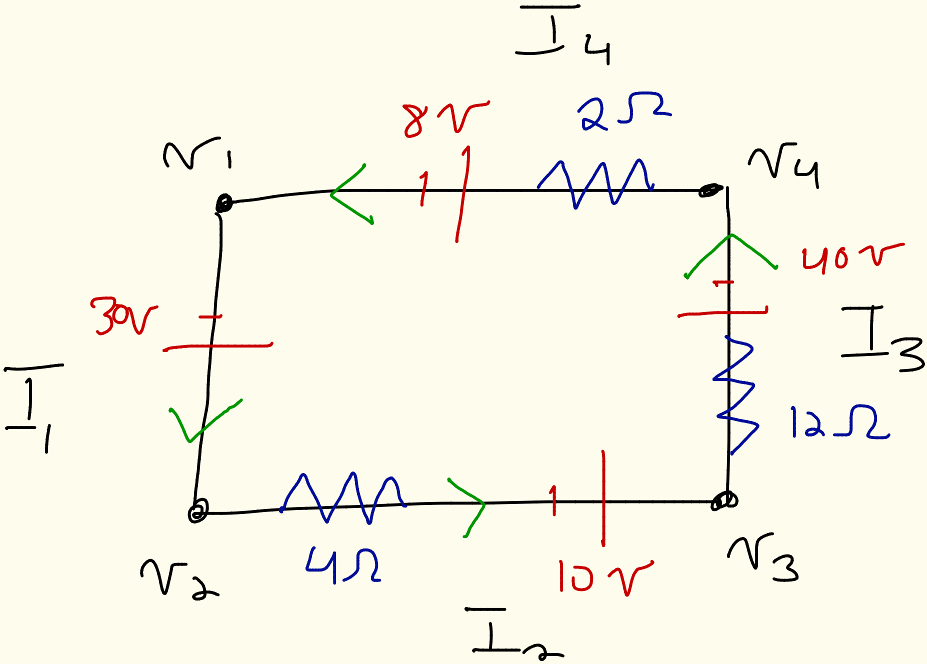
$$I_2 = 306/53$$

$$I_3 = 216/53$$

$$v_1 - v_2 = -120/53$$

we can only solve for v_1 in terms of v_2, but this makes sense.

Another Example



$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

$$A^t = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$b_1 = 30$, $b_2 = 10$, $b_3 = -40$, $b_4 = -8$

Row reduce

$$\left[\begin{array}{cc|c} R & A & \begin{matrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{matrix} \\ A^t & (\text{zeros}) & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix} \end{array}\right]$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 30 \\ 0 & 4 & 0 & 0 & 0 & -1 & 1 & 0 & 10 \\ 0 & 0 & 12 & 0 & 0 & 0 & -1 & 1 & -40 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & -1 & -8 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

rref is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4/9 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -4/9 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -4/9 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -4/9 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -64/9 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 206/9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 104/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{\begin{aligned} &\text{So } I_1 = I_2 = I_3 = I_4 = -4/9 \\ &v_1 = -64/9 + v_4 \\ &v_2 = 206/9 + v_4 \\ &v_3 = 104/3 + v_4 \end{aligned}}$$